Asymptotically Dense Spherical Codes—Part I: Wrapped Spherical Codes

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***Abstract*—A new class of spherical codes called *wrapped spherical codes* is constructed by "wrapping" any sphere packing Λ in Euclidean space onto a finite subset of the unit sphere in one higher dimension. The mapping preserves much of the structure of Λ, and unlike previously proposed maps, the density of wrapped spherical codes approaches the density of Λ as the minimum distance approaches zero. We show that this implies that the asymptotically maximum spherical coding density is achieved by wrapped spherical codes whenever Λ is the densest possible sphere packing.**

***Index Terms*—Asymptotic density, laminated lattices, packing, source and channel coding, spherical codes.**

I. Introduction

A. Overview

THIS paper is the first of two parts that present new spherical code constructions that are asymptotically dense as the minimum distance tends to zero. Part I introduces wrapped spherical codes, which are constructed by mapping finite subsets of sphere packings to the unit sphere in one higher dimension. The precise mapping is carefully chosen to induce good asymptotic density properties. In fact, using our construction with the densest possible sphere packings gives rise to the densest possible spherical codes, asymptotically.

The sequel (Part II, see [1]) to this paper introduces laminated spherical codes, which are constructed by projecting onto the unit sphere a union of known concentric spherical codes of one less dimension. The construction is analogous to the construction of laminated lattices. In fact, asymptotically as the minimum distance tends to zero, the density of laminated spherical codes approaches the density of a laminated lattice packing in one lower dimension.

For large dimensions, the wrapped spherical codes generally outperform the laminated spherical codes, whereas the laminated codes perform better for low dimensions such as three and four. Both families of spherical codes are asymptotically superior to existing spherical codes. Also, wrapped spherical codes can be constructed in any dimensional Euclidean space, whereas laminated spherical codes have been constructed for dimensions less than 50. The wrapped codes have a somewhat simpler construction than the laminated spherical codes.

In the remainder of Section I, basic definitions are given. Section II discusses the best known upper and lower bounds on spherical code density. In Section III, we give the formal construction of wrapped spherical codes and compute their asymptotic density. Proofs of two of the technical lemmas can be found in Appendices I and II. Numerical performance of wrapped codes is presented in Part II [1], in order to compare the wrapped spherical codes with laminated spherical codes.

B. Preliminary Definitions

A k-dimensional *spherical code* is a set of points in $\mathbb{R}^k$ that lie on the surface of a k-dimensional unit radius sphere. Some of the many applications of spherical codes include signaling on a Gaussian channel with equal energy signal sets [2], [3]; spherical vector quantization, used in low bit-rate speech coding and other source-coding problems [4], [5]; efficient searches of k-dimensional space [6]; numerical evaluation of integrals on spheres [7]; and the computation of the minimum energy configuration of point charges on a sphere [8] for chemistry and physics applications. In this paper, we concentrate on the generic spherical code design problem (with respect to minimum distance), rather than a particular application of spherical codes.

Denote the surface of the unit radius k-dimensional Euclidean sphere by[1]

$$\Omega_k \equiv \left\{(x_1, \cdots, x_k) \in \mathbb{R}^k : \sum_{i=1}^{k} x_i^2 = 1\right\}, \qquad (1)$$

the $(k-1)$-dimensional content (surface area) of Ω_k by $S_k = (k\pi^{k/2})/\Gamma((k/2)+1)$, and the k-dimensional content (volume) of Ω_k by $V_k = \pi^{k/2}/\Gamma((k/2)+1)$, where Γ is the usual gamma function defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt.$$

The *minimum distance* of a k-dimensional spherical code $\mathcal{C} \subset \Omega_k$ is defined as

$$d \equiv \min_{\substack{X, Y \in \mathcal{C} \\ X \neq Y}} \|X - Y\| \qquad (2)$$

[1] The notation for the surface of the unit k-dimensional sphere varies somewhat in the literature, and Ω_k [9]–[15], S_k [16], and S^{k-1} [17]–[19], have all been used.

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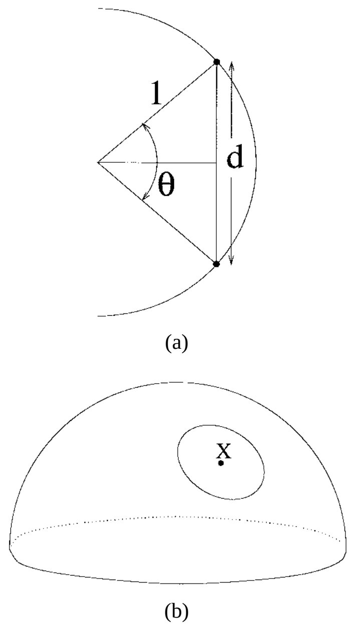


Fig. 1. (a) The relationship between minimum distance d and minimum separating angle θ. (b) A spherical cap.

where $\|\cdot\|$ is the Euclidean distance in $\mathbb{R}^k$. The minimum distance of a spherical code is directly related to the "quality" of the code in many channel coding applications. For channels with high signal-to-noise ratios, one generally desires to maximize the minimum distance for a given number of codepoints in a code.

As this paper concentrates on asymptotically small d, it is important to clarify some notation. For a function $g(d)$, let $O(g(d))$ denote any function $f(d)$ for which there exists positive constants c and d_0 such that $0 \le f(d) \le cg(d)$ for all $d \in (0, d_0)$. Requiring $f(d) \ge 0$ is not the standard usage of $O(\cdot)$, although it has also been used in the computational complexity literature [20]. This usage simplifies the presentations of several bounds in the paper. Note that with this definition, $f(d) = O(g(d))$ and $f(d) = -O(g(d))$ cannot both be true unless $f(d) = 0$ for all d in some interval $(0, d_0)$. Also, for example, $f(d) \ge g(d) + O(d)$ is equivalent to $f(d) \ge g(d)$ for sufficiently small d. The dimension k will be regarded as a constant in the asymptotic analysis.

The *angular separation* between two points (vectors) $X, Y \in \Omega_k$ is $\cos^{-1}(X \cdot Y)$. The *minimum angular separation* of spherical code $\mathcal{C}$ (see Fig. 1(a)) is defined as

$$\theta \equiv 2\sin^{-1}(d/2) \tag{3}$$
$$= d + \frac{d^3}{24} + O(d^5). \tag{4}$$

The set of points on Ω_k, whose angular separation from a fixed point $X \in \Omega_k$ is less than ϕ, is called a *spherical cap centered at X with angular radius ϕ* and is denoted by

$$c_X(k, \phi) \equiv \{Y \in \Omega_k : X \cdot Y > \cos\phi\}. \tag{5}$$

This is illustrated in Fig. 1(b). When the center X of a spherical cap is not relevant, the notation may be abbreviated as $c(k, \phi)$. If spherical caps of angular radius $\theta/2$ are centered at the codepoints of a spherical code with minimum distance d and minimum angular separation θ, then the caps are disjoint. The $(k-1)$-dimensional content of $c(k, \theta/2)$ is given by

$$\begin{aligned} &S(c(k, \theta/2)) \\ &\quad= S_{k-1}\int_0^{\theta/2} \sin^{k-2} x\, dx \\ &\quad= S_{k-1}\int_0^{\theta/2} (x - x^3/6 + O(x^5))^{k-2}\, dx \\ &\quad= S_{k-1}\int_0^{\theta/2} \left(x^{k-2} - \frac{k-2}{6}\, x^k \pm O(x^{k+2})\right) dx \\ &\quad= S_{k-1}\left(\frac{1}{k-1}\,(\theta/2)^{k-1} - \frac{k-2}{6(k+1)}\,(\theta/2)^{k+1} \right. \\ &\qquad\qquad \left. \pm O(\theta^{k+3})\right) \qquad (6) \\ &\quad= V_{k-1}(\theta/2)^{k-1} - O(\theta^{k+1}) \qquad (7) \\ &\quad= V_{k-1}\left(\frac{d}{2}\right)^{k-1} + O(d^{k+1}) \qquad (8) \end{aligned}$$

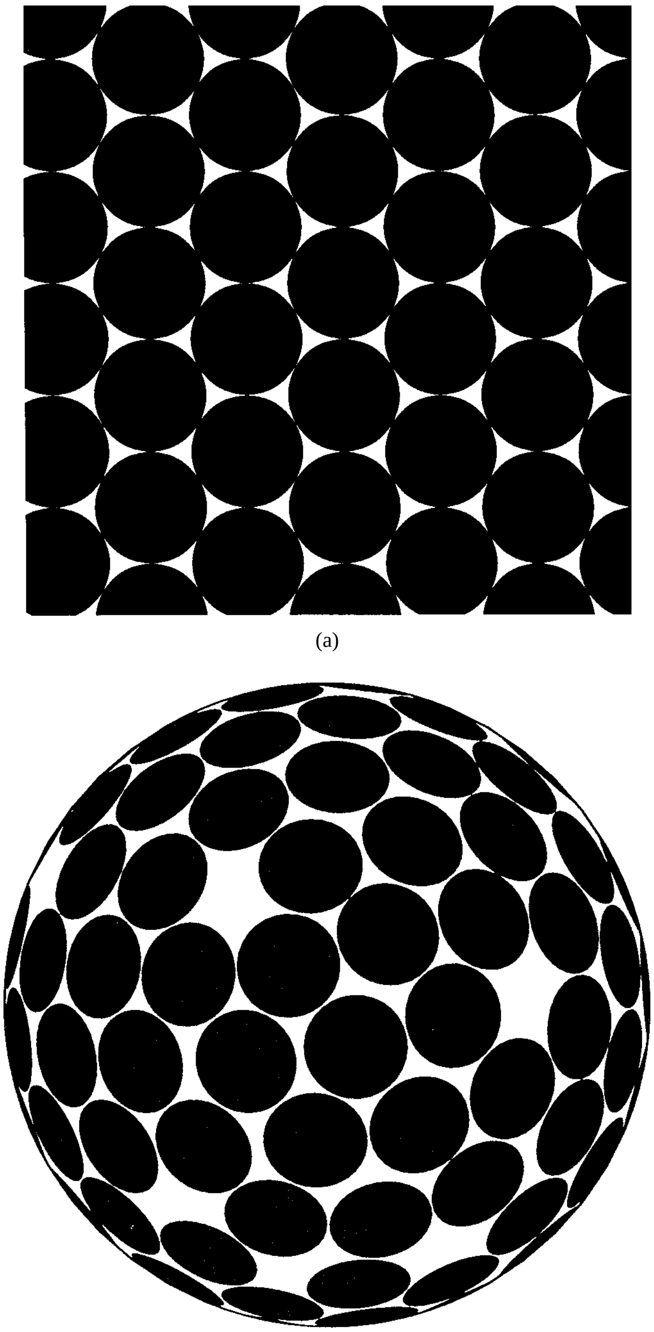
where (8) follows by using (4).

A *sphere packing* (or simply *packing*) is a set of mutually disjoint, equal-radius, open spheres. The *packing radius* is the radius of the spheres in a packing. As defined in [21], a packing is said to have *density* Δ if the ratio of the volume of the part of a hypercube covered by the spheres of the packing to the volume of the whole hypercube tends to the limit Δ, as the side of the hypercube tends to infinity. That is, the density is the fraction of space occupied by the spheres of the packing. The *density $\Delta_{\mathcal{C}}$ of a spherical code $\mathcal{C} \subset \Omega_k$ with minimum distance d* is the ratio of the total $(k-1)$-dimensional content of $|\mathcal{C}|$ disjoint spherical caps centered at the codepoints and with angular radius $\theta/2$, to the $(k-1)$-dimensional content of Ω_k; that is, $\Delta_{\mathcal{C}} \equiv |\mathcal{C}| \cdot S(c(k, \theta/2))/S_k$. This definition is analogous to the definition of the density of a sphere packing (see Fig. 2). Let $M(k, d)$ be the maximum cardinality of a k-dimensional spherical code with minimum distance d, and let $\Delta(k, d)$ be the maximum density among all k-dimensional spherical codes with minimum distance d. Then

$$\Delta(k, d) \equiv \frac{M(k, d)S(c(k, \theta/2))}{S_k}. \tag{9}$$

The value of $M(k, d)$ is easy to compute for all d when $k = 2$. However, $M(k, d)$ is unknown for all $k \ge 3$ except for a handful of values of d, although a number of bounds have been given [9], [11], [12], [17], [22]–[31]. For asymptotically small d, the tightest known upper bounds on $M(k, d)$ are given in [25] for $k = 3$ and in [22] for $k \ge 4$, and a code construction in [28] provides the tightest known lower bound. However, there exists a nonvanishing gap between these upper and lower bounds as $d \to 0$.

A family of codes $\{\mathcal{C}(k, d)\}$ is *asymptotically optimal* if $|\mathcal{C}(k, d)|/M(k, d) \to 1$ as $d \to 0$, or equivalently, if $\Delta_{\mathcal{C}(k,d)}/\Delta(k, d) \to 1$ as $d \to 0$. Given a densest packing



(a)

(b)

Fig. 2. Sphere packing density and spherical code density. (a) The sphere packing density is the percentage of the square that is shaded, as the length of the side of the square goes to infinity. (b) The spherical code density is the percentage of the unit sphere that is shaded.

in $\mathbb{R}^{k-1}$, we show how to construct asymptotically optimal spherical codes. Fig. 3 shows the asymptotic densities of the best spherical codes, i.e., the densities of the spherical codes as $d \rightarrow 0$, for dimensions up to 50. For each dimension, the limiting density of spherical codes constructed by the various methods is computed. To emphasize the comparison to wrapped spherical codes, this limiting density is divided by the asymptotic density achieved by the wrapped spherical codes. Hence, in Fig. 3, the normalized density of the wrapped code is identically 1, while the normalized density of any code whose asymptotic density is worse than the wrapped code is less than 1.

II. Bounds on the Density of a Spherical Code

When designing for minimum distance, the best k-dimensional spherical code with minimum distance d is one which has the largest number of codepoints, namely, $M(k, d)$. Consequently, most previous authors have used $M(k, d)$ as the figure of merit for a spherical code. Using (9), any bound on the code size $M(k, d)$ may be converted to a bound on the code density $\Delta(k, d)$. Whereas the code size increases without bound as d becomes small, the density is always a number in the interval $[0, 1]$. Therefore, for small minimum distances the bounds are more easily compared if they are expressed in terms of density. Additionally, using density as the figure of merit instead of code size allows one to compare the quality of codes with different minimum distances. Conversion of the bounds from statements about $M(k, d)$ to statements about $\Delta(k, d)$ also highlights the gap between the existing upper and lower bounds, and brings to light the fact that some of the best bounds known are not asymptotically tight.

A. Asymptotic Spherical Code Density

Intuitively, as $d \rightarrow 0$, the density of the densest k-dimensional spherical code approaches that of the densest sphere packing in $k-1$ dimensions.[2] Let the *asymptotically maximum spherical coding density* be defined by

$$\Delta_k^{\text{code}} \equiv \lim_{d \rightarrow 0} \Delta(k, d)$$

where $\Delta(k, d)$ is the maximum density of a k-dimensional spherical code with minimum distance d, as defined in (9). Let Δ_k^{pack} denote the density of the densest k-dimensional sphere packing.

Observation 1 $\Delta_k^{\text{code}} = \Delta_{k-1}^{\text{pack}}$.

A formal proof of this observation may be found in [4]. Some justification for this observation can be seen by the fact that Ω_k may be approximated by convex polytopes, as shown in Fig. 4 when $k = 3$. On each face of the polytope, a $(k-1)$-dimensional sphere packing may be placed.

The wrapped spherical codes presented in this paper give efficient constructions based on the intuition of Observation 1. In addition to being asymptotically optimal, they also outperform other codes for moderate sizes of d. This is demonstrated numerically in [1, figs. 6–8].

Note that since $\Delta_2^{\text{pack}} = \pi/(2\sqrt{3})$, the maximum asymptotic density possible for a three-dimensional spherical code is $\pi/(2\sqrt{3})$. The densest sphere packing is not known for $k > 2$, however.[3] From Observation 1, upper bounds on asymptotic sphere packing densities give upper bounds on spherical code densities. For example, Rogers's bound [21] on sphere packing densities is used to provide the upper bound in Fig. 3.

[2] For $k = 3$, a densest covering of Earth with dimes looks like the hexagonal lattice packing A_2 to someone standing on the Earth.

[3] In 1991, Hsiang announced a proof [32] (later published in [33]) of Kepler's conjecture, dating back to 1611, that the face-centered cubic packing is the densest packing in three dimensions. However, the validity of the proof has been questioned [34]. Hsiang has published a rejoinder [35].

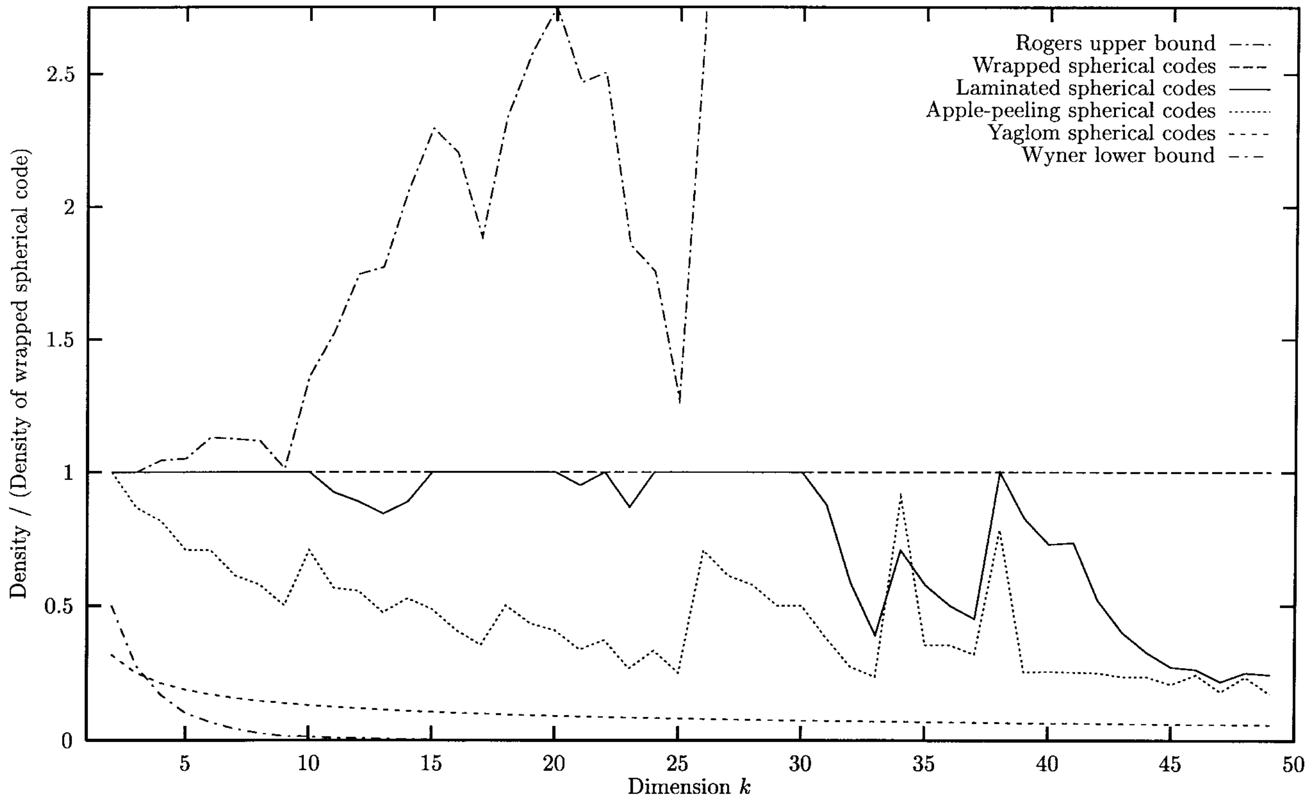


Fig. 3. The ratio of the asymptotic density of various spherical codes to that of wrapped spherical codes constructed from the densest known packings, as a function of dimension. Except for the upper bound, all curves are below 1, indicating the superiority of the wrapped spherical code.

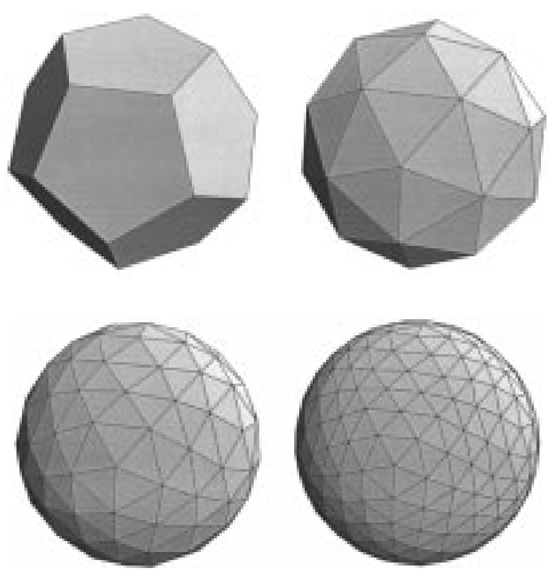


Fig. 4. The unit sphere may be approximated by polytopes.

B. Upper Bounds on Density

1) Fejes Tóth Upper Bound ($k=3$): For small d, the smallest known upper bound on $M(3, d)$ is given by Fejes Tóth [25], who proved that disjoint spherical caps with angular radius $\theta/2$ cannot be packed on the sphere Ω_3 in a denser configuration than that of three mutually tangent spherical caps with angular radius $\theta/2$ (see Fig. 5). As a result, the minimal angular separation was shown to be bounded as

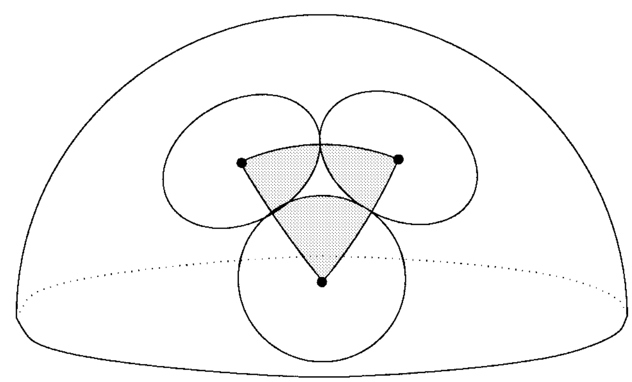


Fig. 5. The density of a spherical code with minimal angle separation θ is at most the percentage of area covered in the spherical triangle formed by three mutually touching caps of angular radius $\theta/2$.

$$\theta \leq \cos^{-1}\left[\frac{\cot^2\left(\dfrac{M(3,\,d)\pi}{6M(3,\,d)-12}\right)-1}{2}\right] \tag{10}$$

from which the following lemma is obtained.

Lemma 1: The density $\Delta_{\mathcal{C}}$ of any three-dimensional spherical code $\mathcal{C}$ with minimum distance $d>0$ satisfies $\Delta_{\mathcal{C}} \leq \pi/(2\sqrt{3})$.

Proof: Combining (3) and (10) gives

$$d \leq \sqrt{3-\cot^2\left(\frac{\pi}{6-12/M(3,d)}\right)}$$

or, equivalently,

$$M(3, d) \le 2 \cdot \left[1 - \frac{\pi/6}{\cot^{-1}\sqrt{3-d^2}}\right]^{-1}.$$

Using

$$S(c(3, \theta/2)) = S_2 \int_0^{\theta/2} \sin x\, dx = 2\pi(1-\cos\theta/2) = 2\pi\left(1-\sqrt{1-d^2/4}\right)$$

and (9),

$$\Delta(3, d) \le \frac{2 \cdot \left[1 - \frac{\pi/6}{\cot^{-1}\sqrt{3-d^2}}\right]^{-1} \cdot 2\pi(1-\sqrt{1-d^2/4})}{4\pi} = \frac{1-\sqrt{1-d^2/4}}{1-\frac{\pi/6}{\cot^{-1}\sqrt{3-d^2}}}. \quad (11)$$

Some elementary (but laborious) calculus reveals that (11) is monotonically decreasing for $d \in (0, 1)$, and thus approaches its supremum as $d \to 0$. The limit of (11) as $d \to 0$ is $\pi/(2\sqrt{3})$. □

Since $\Delta_2^{\text{pack}} = \pi/(2\sqrt{3})$, Lemma 1 agrees with Observation 1, and implies the following.

Corollary 1: The Fejes Tóth upper bound on spherical code size is asymptotically tight.

Proof: From Observation 1, it follows that

$$\Delta_3^{\text{code}} = \Delta_2^{\text{pack}} = \pi/(2\sqrt{3}).$$

The Fejes Tóth bound implies

$$\Delta_3^{\text{code}} = \lim_{d\to 0} \Delta(3, d) = \pi/(2\sqrt{3}).$$ □

2) Coxeter Upper Bound ($k \ge 4$): Böröczky [36] proved that in a k-dimensional space of constant curvature, the density of a packing of equal radii k-dimensional spheres cannot exceed the density of $k+1$ such spheres that mutually touch one another. This verified Coxeter's conjecture [22] that spherical caps on Ω_k can be packed no denser than k spherical caps on Ω_k that simultaneously touch one another. The centers of these caps lie on the vertices of a regular spherical simplex. This is a generalization of the Fejes Tóth bound for $k = 3$. Coxeter's bound is given by

$$M(k, d) \le \frac{2F_{k-1}(\alpha)}{F_k(\alpha)}$$

where α is determined by

$$\sec 2\alpha = \frac{2}{2-d^2} + k - 2$$

and where $F_k : \mathbb{R} \to \mathbb{R}$ is Schlafli's function [10] given by the recursive relation

$$F_k(\alpha) = \frac{2}{\pi}\int_{\sec^{-1}((k-1)/2)}^{\alpha} F_{k-2}(\beta)\, d\theta$$

with $\sec 2\beta = (\sec 2\theta) - 2$, and the initial conditions $F_0(\alpha) = F_1(\alpha) = 1$. Unfortunately, the computational complexity of evaluating the bound is high for $k > 3$, as it involves $\lfloor k/2 \rfloor$ nested integrals.

Coxeter's bound was motivated by an argument of Rogers, who showed that the fraction of a simplex's volume covered by spheres centered on the simplex vertices is an upper bound on the density of a sphere packing, despite the fact that simplices cannot tile $\mathbb{R}^k$ for $k \ge 3$ [21]. Coxeter's bound uses a similar argument, except that $\mathbb{R}^k$ is replaced by Ω_k and simplices are replaced by spherical simplices. (A similar argument was used for a proposed bound on the quantization coefficient in $\mathbb{R}^k$ [37].) Thus when Coxeter's bound on $M(k, d)$ is translated to a bound on $\Delta(k, d)$, it becomes a statement about the density of a spherical simplex of edge-length d. As $d \to 0$, the density of the spherical simplex of edge-length d on Ω_k approaches the density of a regular simplex in $\mathbb{R}^{k-1}$, which gives us the following lemma.

Lemma 2: Coxeter's upper bound on asymptotic spherical coding density Δ_k^{code} equals Rogers's upper bound on sphere packing density $\Delta_{k-1}^{\text{pack}}$.

Corollary 2: Coxeter's upper bound on Δ_k^{code} is tight if and only if Rogers's upper bound on $\Delta_{k-1}^{\text{pack}}$ is tight. In particular, Coxeter's bound is not asymptotically tight for $k = 4$.

Proof: The first statement follows from Observation 1 and Lemma 2. The second statement follows because Rogers's bound of $\Delta_3^{\text{pack}} \le 0.7796$ has been improved to $\Delta_3^{\text{pack}} \le 0.7784$ [38]. □

C. Apple-Peeling Spherical Codes and Other Lower Bounds on Density

Any known k-dimensional spherical code with minimum distance d gives a lower bound on $M(k, d)$, and hence on $\Delta(k, d)$. Much work has been done to find the best spherical codes, such as from binary codes [12], [13], [31]; shells of lattices [15], [29], [39]; permutations of a set of initial vectors [40], [41]; simulated annealing or repulsion-energy methods [8], [28], [42]; concatenations of lower dimensional codes [30]; projections of lower dimensional objects [24], [26], [28]; and other means [2], [3], [27], [43]–[46].

Unfortunately, none of the spherical coding methods above performs well in a fixed dimension k, as $d \to 0$. Also, many of the methods above produce spherical codes for only a finite number of minimum distances d. Previously, the best spherical codes known for asymptotically small d were the so-called apple-peeling codes due to El Gamal *et al.* [28]. Their technique resembles peeling an apple in three dimensions, and is described below for comparison purposes. We also compute the asymptotic density of the apple-peeling codes.

Let $\mathcal{C}^*(k-1, d)$ denote any $(k-1)$-dimensional spherical code with minimum distance d, whose codepoints are indexed from 1 to $|\mathcal{C}^*(k-1, d)|$. The *apple-peeling spherical code* $\mathcal{C}^A(k, d)$ on Ω_k with respect to $\mathcal{C}^*(k-1, \cdot)$ is defined in [28] as the set of points

$$\{(x_1(i, j)\cos\eta(i), \cdots, x_{k-1}(i, j)\cos\eta(i), \sin(\eta(i)))\}$$

such that

$$i \in \{l \in \mathbb{Z} : -\pi/2 \le \eta(l) \le \pi/2\} \quad (12)$$
$$j \in \{1, \cdots, |\mathcal{C}^*(k-1, d/\cos\eta(i))|\}$$
$$\eta(i) \equiv (i+1/2)\theta$$
$$X(i, j) = (x_1(i, j), \cdots, x_{k-1}(i, j)) \text{ is the } j\text{th codeword of } \mathcal{C}^*(k-1, d/\cos\eta(i)). \quad (13)$$

It is verified in [28] that the apple-peeling code has minimum distance d.

Summing over all values of i in (12) and choosing $\mathcal{C}^*(k-1, d/\cos\eta(i))$ to be a maximum size code for all i give the lower bound

$$M(k, d) \ge 2 \cdot \sum_{i=0}^{\lfloor(\pi/2\theta)-(1/2)\rfloor} M(k-1, d/\cos\eta(i)). \quad (14)$$

The Beta function $\beta(\cdot, \cdot) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is used in the following lemma.

Lemma 3: The density of the densest k-dimensional apple-peeling spherical code $\mathcal{C}^A(k, d)$ approaches

$$\tfrac{1}{2} \cdot \Delta_{k-2}^{\text{pack}} \beta(\tfrac{k}{2}, \tfrac{1}{2})$$

as $d \to 0$.

Proof: See Appendix I.

Corollary 3: For all $k \ge 3$, the k-dimensional apple-peeling code is not asymptotically optimal. Furthermore, the ratio of the asymptotic density of apple-peeling codes in k-dimensions to the maximum asymptotic spherical coding density $\Delta(k)$ tends to 0 as $k \to \infty$.

Proof: This follows from the facts that $\frac{1}{2} \cdot \beta(\frac{k}{2}, \frac{1}{2})$ is less than 1 for all $k \ge 3$ and that this quantity approaches 0 as k tends to ∞. □

The numerical values of $M(k-1, d)$ for $k \ge 4$ are not presently known, except for a handful of values of d; hence, $\Delta_{\mathcal{C}^A(k,d)}$ cannot be easily evaluated using (14). Also, $\Delta_{k-2}^{\text{pack}}$ is not known for $k \ge 5$, and so the asymptotic performance is also difficult to evaluate. However, the numerical values of the asymptotic density of the best realizable apple-peeling codes, given the current state of knowledge of $M(k-1, d)$, can be determined as follows. If, in (13), $\mathcal{C}^*(k-1, d/\cos\eta(i))$ is chosen to be the best code known with the given parameters, then a lower bound on $\Delta(k, d)$ can be computed. By Observation 1, the density of $\mathcal{C}(k-1, d/\cos\eta(i))$ can be as high as the density of the best sphere packing known in $\mathbb{R}^{k-2}$. The asymptotic densities of the best apple-peeling codes currently realizable are given by replacing $\Delta_{k-2}^{\text{pack}}$ in the formula for the density in Lemma 3 with the density of the best sphere packing *known* in $\mathbb{R}^{k-2}$. This apple-peeling code asymptotic density is shown in Fig. 3, using a table [10] of the best sphere packings known, along with recent improvements from [47] and [48]. The asymptotic density of the wrapped spherical codes is equal or higher in every dimension.

III. Construction of Wrapped Spherical Codes

In this section, a mapping is introduced which effectively "wraps" any packing in $\mathbb{R}^{k-1}$ around Ω_k; hence, the spherical codes it constructs are referred to as wrapped spherical codes. This technique creates codes of any size and thus provides a lower bound on achievable minimum distance as a function of code size. We will show that the spherical code density approaches the density of the underlying packing, as $d \to 0$.

Any spherical code can be described by the projection of its codepoints to the interior of a sphere of one less dimension via the mapping

$$\left(x_1, \cdots, x_{k-1}, \sqrt{1 - \sum_{i=1}^{k-1} x_i^2}\right) \to (x_1, \cdots, x_{k-1}).$$

Conversely, a k-dimensional spherical code may be obtained by placing codepoints within Ω_{k-1} and projecting each codepoint onto Ω_k using the reverse mapping. This mapping was used by Yaglom [24] to map a $(k-1)$-dimensional lattice Λ onto Ω_k. However, the distortion created by mapping Λ to Ω_k gives poor asymptotic spherical code densities, even if Λ is the densest lattice in $k-1$ dimensions, as summarized in Fig. 3. This is due to the "warping" effect on the codepoints near the boundary, as illustrated in Fig. 6.

Intuitively, *any* continuous mapping will have this distortion problem—e.g., it is difficult to wrap a piece of paper around a ball without wrinkles forming somewhere on the ball. Wrapped spherical codes avoid the distortion problem by partitioning Ω_k into annuli, as shown in Fig. 7(a). Within each annulus, a continuous, small-distortion mapping is used, as shown in Fig. 7(b); this mapping is similar but not identical to the mapping Yaglom used. Using the intuition of Observation 1, namely, that small patches of Ω_k can be approximated by small patches of $\mathbb{R}^{k-1}$, each annulus is mapped to a finite region in $\mathbb{R}^{k-1}$. A dense sphere packing is placed in that finite region and the inverse mapping is applied. The resulting points lie on the original annuli and are the codepoints of the spherical code.

There is some flexibility in choosing the mapping f from each annulus to $\mathbb{R}^{k-1}$, but there are several desirable characteristics one would like f to have. First, within each annulus, the mapping should be continuous. This will guarantee that points of an annulus which are close together remain close together after f is applied to them. One also would like f to be an easily computable one-to-one function whose inverse is also easily computable. Third, the mapping should have the property that if X and Y belong to the same annulus of Ω_k, then $\|f(X) - f(Y)\| \le \|X - Y\|$. If this is the case, then a sphere packing with packing radius d in $\mathbb{R}^{k-1}$ can be projected to the ith annulus of Ω_k via f^{-1}, to result in a spherical code with minimum distance *at least* d. Finally, f should have the property that the $(k-1)$-dimensional content of an annulus should be very similar to the $(k-1)$-dimensional content of the image of that annulus under f. This is necessary for the asymptotic density of the spherical code to equal the density of the sphere packing used in $\mathbb{R}^{k-1}$.

We now discuss the wrapped spherical code construction in more detail. In particular, we give a precise description of our choice of the function f that satisfies the above requirements. Let Λ denote a sphere packing in $\mathbb{R}^{k-1}$ which has minimum distance d and density Δ_Λ. Λ may be either a lattice packing or a nonlattice packing. We define the *latitude* of a point $X =$

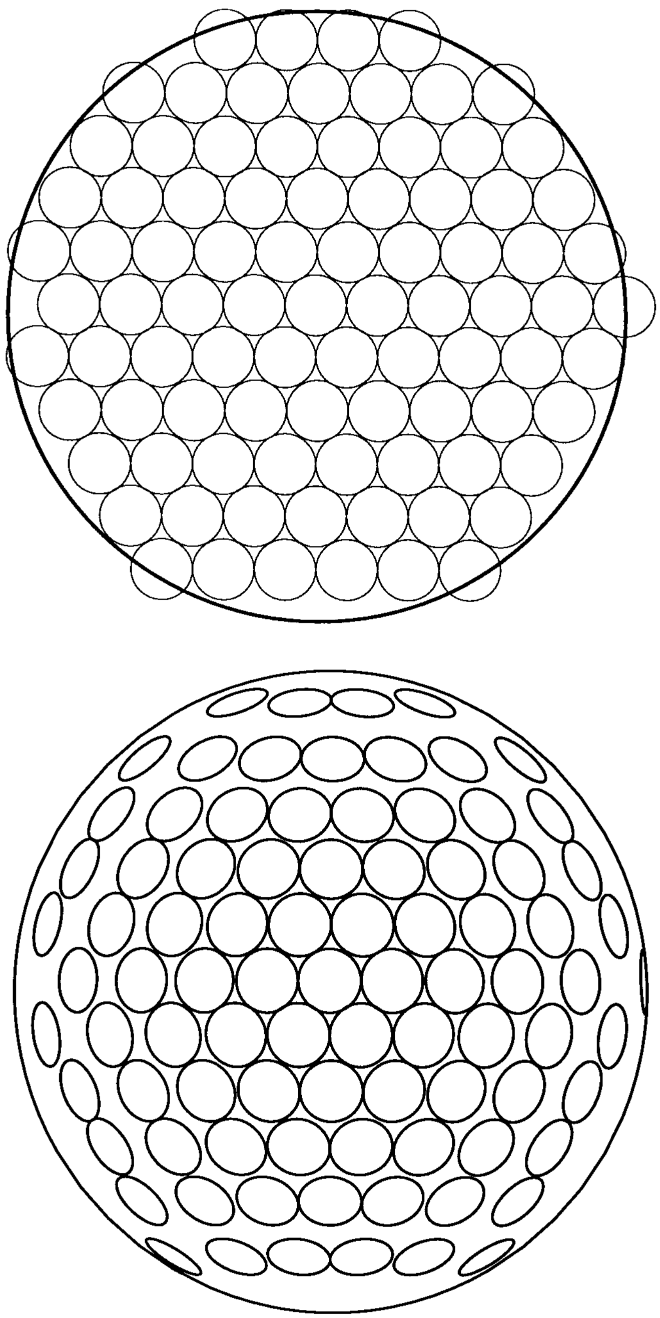
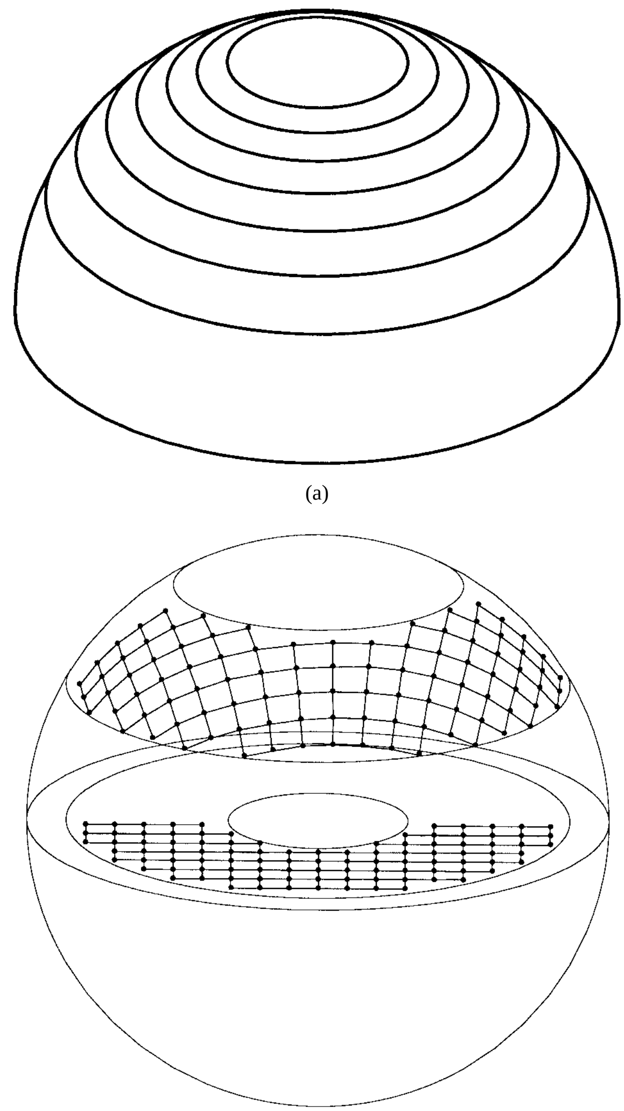


Fig. 6. Yaglom's mapping consists of taking the part of a lattice within Ω_{k-1}, shown at the top, and projecting it out of the page onto the surface of Ω_k, shown at the bottom.

$(x_1, \cdots, x_k) \in \Omega_k$ as $\sin^{-1}(x_k)$, i.e., the angle subtended from the "equator" to X. Let $-\pi/2 = \alpha_0 < \cdots < \alpha_N = \pi/2$ be a sequence of latitudes of annulus boundaries. The *ith annulus* is defined as the set of points $(x_1, \cdots, x_k) \in \Omega_k$ that satisfy $\alpha_i \le \sin^{-1} x_k < \alpha_{i+1}$ (i.e., points between consecutive latitudes), and it is denoted by A_i. The real numbers $\alpha_0, \cdots, \alpha_N$ will be chosen later to yield a large code size.

For each i, we define a one-to-one mapping f_i from A_i to a subset of $\mathbb{R}^{k-1}$. For each $X = (x_1, \cdots, x_k) \in A_i$, let

$$X_L = \arg\min_Z \{\|X - Z\| : Z = (z_1, \cdots, z_{k-1}, \sin \alpha_i) \in \Omega_k\}$$



(a)

(b)

Fig. 7. (a) Annuli of Ω_3. (b) Relationship between Z^2 lattice below and points of the induced wrapped spherical code above, in an annulus.

i.e., the closest point to X that lies on the border between A_{i-1} and A_i. This is shown in Fig. 8. Let prime notation denote the mapping from $\mathbb{R}^k$ to $\mathbb{R}^{k-1}$ obtained by the deletion of the last coordinate, so that, for example, $X' = (x_1, \cdots, x_{k-1})$. The function f_i is defined by

$$f_i(X) \equiv \frac{X'}{\|X'\|} \cdot (\|(X_L)'\| - \|X_L - X\|)_+ \qquad (15)$$

where $(x)_+ \equiv \max(0, x)$. Define the *ith buffer region* of Ω_k as the set

$$B_i \equiv \{X \in A_i : \|X - X_L\| < d\}.$$

The *wrapped spherical code* $\mathcal{C}_W$ with respect to a packing Λ having minimum distance d is defined as

$$\mathcal{C}_W \equiv \bigcup_i f_i^{-1}(\Lambda \setminus \{0\}) \setminus B.$$

An example of a wrapped spherical code is shown in Fig. 9.

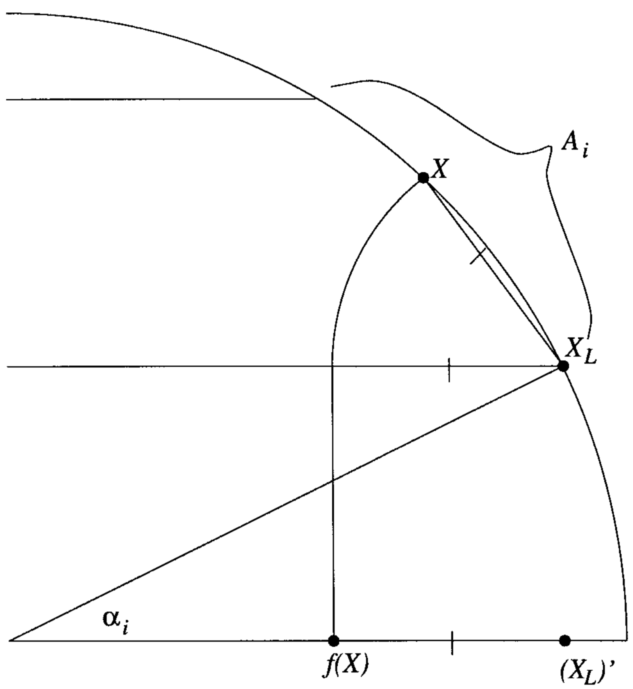


Fig. 8. Geometrical interpretation of the mapping f_i.

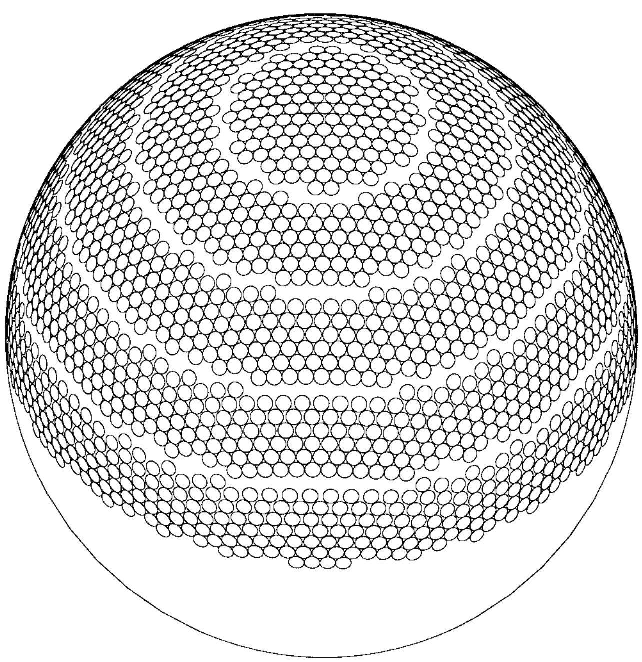


Fig. 9. The wrapped spherical code $\mathcal{C}_W^{\Lambda_2}(3, 0.05)$.

Remarks:

- f_i is a continuous and one-to-one function of A_i.
- Both f_i and f_i^{-1} are easily computed, as they are merely scalings of their input and the addition or deletion of the last coordinate.
- The image of the annulus A_i in Ω_k under f_i is a region in $\mathbb{R}^{k-1}$ bounded by two concentric $(k-1)$-dimensional spheres in $\mathbb{R}^{k-1}$.
- Annulus outer boundaries are projected vertically downward. That is, a point $X = (x_1, \cdots, x_k)$ lying on the boundary between A_{i-1} and A_i (i.e., such that $x_k = \sin^{-1} \alpha_i$), has the property that $f(X) = (x_1, \cdots, x_{k-1})$. Annulus inner boundaries are projected vertically downward and are then contracted.

As this paper is chiefly concerned with asymptotic performance, discussion on small codebook improvements possible for moderately large minimum distances will be limited. A number of simple improvements are possible. One such improvement involves the buffer region B, which is included in the code definition solely to ensure the minimum distance requirement is met. For a particular value of d, a careful choice of latitudes $\{\alpha_i\}$ may make much of the buffer region unnecessary.

As desired, f_i has the property that $\|f_i(X) - f_i(Y)\| \leq \|X - Y\|$. Given that $\|f_i(X)\| < \|X\|$ and that the orientation of X is the same as that of $f_i(X)$, this property is somewhat intuitive. A rigorous proof is somewhat cumbersome, however, and is, therefore, relegated to Appendix II. We state the result in the following lemma.

Lemma 4: If

$$X = (x_1, \cdots, x_k) \in \Omega_k$$

and

$$Y = (y_1, \cdots, y_k) \in \Omega_k$$

are in the annulus A_i, then

$$\|f_i(X) - f_i(Y)\|^2 \leq \|X - Y\|^2.$$

Proof: See Appendix II.

Corollary 4: If Λ is a sphere packing with minimum distance d, then the minimum distance of the wrapped spherical code $\mathcal{C}_W$ is also at least d.

Proof: If distinct $X, Y \in \mathcal{C}_W$ belong to the ith annulus, then $\|X - Y\| \geq \|f_i(X) - f_i(Y)\| \geq d$, since the minimum distance of Λ is d. If X and Y belong to different annuli, then the definition of B guarantees their separation is at least d. □

IV. Asymptotic Density of the Wrapped Spherical Code

Let $\phi_i = \alpha_{i+1} - \alpha_i$ denote the *angular separation* of the ith annulus, let $\overline{\phi} = \max(\phi_i)$, and let $\underline{\phi} = \min(\phi_i)$. In order to have an asymptotically optimal spherical code, one expects that as $d \to 0$, the angular separation of the annuli must go to zero in order to make the curvature of the annulus negligible, and that, consequently, the number of annuli must increase. As $d \to 0$, we must also be able to fit in more and more points in each annulus in order to make the density within each annulus approximately equal to the maximum packing density. The following theorem proves that these two conditions are sufficient for the wrapped spherical code to be asymptotically optimal.

Theorem 1: Let Λ be a $(k-1)$-dimensional sphere packing with minimum distance d. Let $\mathcal{C}_W$ be a wrapped spherical code with respect to Λ and with latitudes $\alpha_0, \cdots, \alpha_N$. If the maximum and minimum annulus angular separations satisfy $\lim_{d \to 0} [\overline{\phi} + (d/\underline{\phi})] = 0$, then the asymptotic density of $\mathcal{C}_W$ approaches the density of Λ, i.e., $\lim_{d \to 0} \Delta_{\mathcal{C}_W} = \Delta_\Lambda$.

Proof: There are two components needed to compute the density of a spherical code on Ω_k: the total $(k-1)$-dimensional content of the caps in the code, and the $(k-1)$-dimensional content of Ω_k. The density equals the former quantity divided by the latter. To prove the theorem, we show that the $(k-1)$-dimensional content of spheres packed in the images of the annuli converge to the $(k-1)$-dimensional content of the caps in the wrapped spherical code, and that the images themselves have content which converges to that of Ω_k.

We concentrate on an arbitrary annulus, the ith annulus A_i, and its image, $R_i = f_i(A_i)$. Let S_{A_i} and S_{R_i} denote the $(k-1)$-dimensional contents of A_i and R_i, respectively. Note that A_i and R_i depend on d. Throughout this proof, constants encompassed by $O(\cdot)$ notation do not depend on i.

First, we show that S_{A_i} is very close to S_{R_i}. Since A_i is the region bounded between latitudes α_i and α_{i+1}, we have

$$\begin{aligned} S_{A_i} &= S_{k-1} \int_{\alpha_i}^{\alpha_{i+1}} \cos^{k-2} x\, dx \\ &= S_{k-1} \int_{\alpha_i}^{\alpha_{i+1}} [\cos^{k-2} \alpha_i - O(x-\alpha_i)]\, dx \qquad (16) \\ &= S_{k-1}\phi_i \cos^{k-2} \alpha_i - O(\phi_i^2) \end{aligned}$$

where (16) follows from the Taylor expansion of $\cos^{k-2} x$ about α_i. The $(k-1)$-dimensional content of R_i is the difference between the $(k-1)$-dimensional contents of two concentric $(k-1)$-dimensional spheres, namely, a sphere of radius $\cos \alpha_i$ and a sphere of radius $\cos \alpha_i - 2\sin(\phi_i/2)$. Thus

$$\begin{aligned} S_{R_i} &= V_{k-1} \cos^{k-1} \alpha_i - V_{k-1}(\cos \alpha_i - 2\sin(\phi_i/2))^{k-1} \\ &= V_{k-1}[\cos^{k-1} \alpha_i \\ &\qquad - (\cos^{k-1} \alpha_i - (k-1)\phi_i \cos^{k-2} \alpha_i + O(\phi_i^2))] \\ &= S_{k-1}\phi_i \cos^{k-2} \alpha_i + O(\phi_i^2) \end{aligned}$$

and it follows that

$$S_{A_i} - O(\phi_i^2) \le S_{R_i} \le S_{A_i}. \qquad (17)$$

Next, $|\Lambda \cap R_i|$ is computed; this is the number of spheres of the sphere packing Λ that lie within R_i. We have

$$|\Lambda \cap R_i| = \frac{\Delta_\Lambda(S_{R_i} - O(d))}{V_{k-1}(d/2)^{k-1}}$$

where the $O(d)$ term appears because the density of lattice points in R_i may be lower than Δ_Λ in a region within distance d of the boundary of R_i, and the $(k-1)$-dimensional content of this boundary region is $O(d)$. Note that $|\Lambda \cap R_i|$ is a little more than $|\mathcal{C}_W \cap A_i|$ because $\mathcal{C}_W$ deletes any point which lies within the buffer region B_i. We have

$$\begin{aligned} |\mathcal{C}_W \cap A_i| &= |\Lambda \cap R_i \backslash \{0\}| - |\Lambda \cap R_i \cap f_i(B_i) \backslash \{0\}| \\ &= \frac{\Delta_\Lambda(S_{R_i} - O(d))}{V_{k-1}(d/2)^{k-1}} - \frac{\Delta_\Lambda O(d)}{V_{k-1}(d/2)^{k-1}} \\ &= \frac{\Delta_\Lambda(S_{R_i} - O(d))}{V_{k-1}(d/2)^{k-1}}. \end{aligned}$$

Thus

$$\begin{aligned} \Delta_{A_i} &= \frac{|\mathcal{C}_W \cap A_i| S(c(k, \theta/2))}{S_{A_i}} \\ &= \frac{\left[\dfrac{\Delta_\Lambda(S_{R_i} - O(d))}{V_{k-1}(d/2)^{k-1}}\right] V_{k-1}\left(\dfrac{d}{2}\right)^{k-1}(1+O(d^2))}{S_{A_i}} \\ &= \frac{\Delta_\Lambda(S_{R_i} - O(d))\,(1+O(d^2))}{S_{A_i}} \\ &\ge \frac{\Delta_\Lambda(S_{A_i} - O(\phi_i^2 + d))\,(1+O(d^2))}{S_{A_i}} \qquad (18) \\ &= \Delta_\Lambda\left(1 - O\left(\frac{\phi_i^2 + d}{S_{A_i}}\right)\right)(1+O(d^2)) \\ &= \Delta_\Lambda\left(1 - O\left(\phi_i + \frac{d}{\phi_i}\right)\right)(1+O(d^2)) \\ &= \Delta_\Lambda\left(1 - O\left(\overline{\phi} + \frac{d}{\underline{\phi}}\right)\right)(1+O(d^2)) \\ &= \Delta_\Lambda - O\left(\overline{\phi} + \frac{d}{\underline{\phi}}\right) \qquad (19) \end{aligned}$$

where (18) follows from the left-hand inequality of (17). If, instead, the right-hand inequality of (17) is used, we obtain

$$\Delta_{A_i} \le \Delta_\Lambda. \qquad (20)$$

Since (19) and (20) do not depend on i,

$$\Delta_\Lambda - O\left(\overline{\phi} + \frac{d}{\underline{\phi}}\right) \le \Delta_{\mathcal{C}_W} \le \Delta_\Lambda. \qquad (21)$$

Since $\lim_{d\to 0}[\overline{\phi} + (d/\underline{\phi})] = 0$, the theorem follows. □

Equation (19) also suggests a choice of $\{\alpha_i\}$ that will provide a fast rate of convergence, and implies the following corollary.

Corollary 5: Let Λ be a $(k-1)$-dimensional sphere packing with minimum distance d, and let $\mathcal{C}_W$ be a wrapped spherical code with respect to Λ and with latitudes given by $\alpha_i = i\sqrt{d}$ for $0 \le i \le \pi/(2\sqrt{d})$. Then the spherical code density satisfies $|\Delta_{\mathcal{C}_W} - \Delta_\Lambda| \le O(\sqrt{d})$.

Proof: The result follows immediately from (21), since $\overline{\phi} = \underline{\phi} = \sqrt{d}$. □

V. Conclusions

A new technique was presented that constructs wrapped spherical codes in any dimension and with any minimum distance. The construction is performed by defining a map from $\mathbb{R}^{k-1}$ to Ω_k. Although any set of points in $\mathbb{R}^{k-1}$ may be wrapped to Ω_k using our technique, if the densest packing in $\mathbb{R}^{k-1}$ is used the wrapped spherical codes are asymptotically optimal, in the sense that the ratio of the density of the constructed code to the upper bound approaches one as the number of codepoints increases. This demonstrates the tightness of the upper bound in [25], asymptotically, and that previous spherical codes are not asymptotically optimal.

Related techniques for creating spherical codes may prove useful. Another class of spherical codes called "laminated spherical codes" [1] have been created by building up shells of codepoints in $\mathbb{R}^{k-1}$ in such a way that when projected directly onto Ω_k they result in a spherical code of minimum distance d. Asymptotically optimal performance has been achieved in three dimensions, and good asymptotic performance is also achieved in higher dimensions, where the k-dimensional laminated spherical code density approaches the density of the laminated lattice Λ_{k-1}. The question of whether the asymptotic density of the k-dimensional laminated spherical code is optimal is equivalent to the question of whether Λ_{k-1} is the densest sphere packing.

For nonasymptotic codes, an important question in channel decoding and quantization is how to find the nearest codepoint to an arbitrary point in $\mathbb{R}^k$. The decoding complexity of the wrapped spherical codes turns out to be equivalent to the decoding complexity of the underlying lattice; a detailed decoding algorithm and complexity analysis can be found in [4].

Appendix I
Proof of Lemma 3

Let $\gamma \in (0, \pi/2)$ and

$$R \equiv \{(x_1, \ldots, x_k) \in \Omega_k : \sin\gamma < x_k < \sin(\gamma+\theta)\}$$

and let $S(R)$ be the $(k-1)$-dimensional content of R. Then

$$\begin{aligned} S(R) &= S_{k-1}\int_\gamma^{\gamma+\theta} \cos^{k-2} x\,dx \\ &= S_{k-1}\int_\gamma^{\gamma+\theta} (\cos\gamma - O(x-\gamma))^{k-2}\,dx \\ &= S_{k-1}\int_\gamma^{\gamma+\theta} \cos^{k-2}\gamma - O(x-\gamma)\,dx \\ &= S_{k-1}\theta\cos^{k-2}\gamma - O(\theta^2). \end{aligned}$$

Since the kth coordinate of every codepoint in $\mathcal{C}^S(k, d)$ is of the form $\sin[(i+1/2)\theta]$, every codepoint in $\mathcal{C}^S(k, d)\cap R$ has the same kth coordinate, say, $\sin\eta$. Thus

$$|\mathcal{C}^S(k, d)\cap R| = M(k-1, d/\cos\eta). \tag{22}$$

By the definition of $\Delta_{k-1}^{\text{code}}$, given any $\epsilon > 0$ there exists a sufficiently small $d_0 > 0$ such that

$$\frac{M(k-1, d/\cos\eta)S(c(k-1, \sin^{-1}(d/(2\cos\eta))))}{S_{k-1}} \le \Delta_{k-1}^{\text{code}} + \epsilon$$

for all $d \le d_0$. Since

$$\sin^{-1}\left(\frac{d}{2\cos\eta}\right) = \sin^{-1}\left(\frac{\sin(\theta/2)}{\cos\eta}\right) = \frac{\theta}{2\cos\eta} - O(\theta^3)$$

one can apply (7) and obtain

$$S(c(k-1, \sin^{-1}(d/(2\cos\eta)))) = V_{k-2}\left(\frac{\theta}{2\cos\eta}\right)^{k-2} - O(\theta^k). \tag{23}$$

Hence

$$\begin{aligned} \Delta_{\mathcal{C}^S(k,d)} &\le \frac{|\mathcal{C}^S(k,d)\cap R|\cdot S(c(k,\theta/2))}{S(R)} \\ &= \frac{M(k-1, d/\cos\eta)S(c(k,\theta/2))}{S(R)} \\ &\le \frac{(\Delta_{k-1}^{\text{code}}+\epsilon)S_{k-1}S(c(k,\theta/2))}{S(c(k-1,\sin^{-1}(d/(2\cos\eta))))S(R)} \qquad (24) \\ &= \frac{(\Delta_{k-1}^{\text{code}}+\epsilon)\left(V_{k-1}(\theta/2)^{k-1} - O(\theta^{k+1})\right)}{\left(V_{k-2}\left(\frac{\theta}{2\cos\eta}\right)^{k-2} - O(\theta^k)\right)(\theta\cos^{k-2}\gamma - O(\theta^2))} \\ &\le \frac{(\Delta_{k-1}^{\text{code}}+\epsilon)V_{k-1}\left(\frac{\theta}{2}\right)^{k-1}}{\theta(\cos^{k-2}\gamma)V_{k-2}\left(\frac{\theta}{2\cos\eta}\right)^{k-2}(1-O(\theta))(1-O(\theta^2))} \\ &= \frac{(\Delta_{k-1}^{\text{code}}+\epsilon)V_{k-1}}{2V_{k-2}} + O(\theta), \qquad (25) \\ &= \frac{\Delta_{k-2}^{\text{pack}}V_{k-1}}{2V_{k-2}} + O(\epsilon+\theta), \end{aligned}$$

where (24) follows from (23), and where (25) follows from

$$\frac{\cos\gamma}{\cos\eta} = \frac{\cos\gamma}{\cos\gamma - O(\theta)} = 1 + O(\theta).$$

By letting $\epsilon \to 0$ and $\theta \to 0$, the result is obtained. □

Appendix II
Proof of Lemma 4

Let $X = (x_1, \ldots, x_k)$, $Y = (y_1, \ldots, y_k)$, and as before, let prime notation denote the deletion of the kth coordinate. Let

$$\alpha(X) = \|f(X)\| = \|(X_L)'\| - \|X_L - X\|.$$

Then

$$\begin{aligned} &\|f_i(X) - f_i(Y)\|^2 \\ &= \sum_{i=1}^{k-1}\left(\frac{\alpha(X)}{\|X'\|}x_i - \frac{\alpha(Y)}{\|Y'\|}y_i\right)^2 \\ &= \alpha(X)^2 + \alpha(Y)^2 - \frac{\alpha(X)\alpha(Y)}{\|X'\|\cdot\|Y'\|}2X'\cdot Y' \\ &= (\alpha(X)-\alpha(Y))^2 + \frac{\alpha(X)\alpha(Y)}{\|X'\|\cdot\|Y'\|} \\ &\quad\cdot(\|X'-Y'\|^2 - (\|X'\|-\|Y'\|)^2) \qquad (26) \\ &\le (x_k-y_k)^2 + (\|X'\|-\|Y'\|)^2 + \frac{\alpha(X)\alpha(Y)}{\|X'\|\cdot\|Y'\|} \\ &\quad\cdot(\|X'-Y'\|^2 - (\|X'\|-\|Y'\|)^2) \qquad (27) \\ &= (x_k-y_k)^2 + \|X'-Y'\|^2 + \left(\frac{\alpha(X)\alpha(Y)}{\|X'\|\cdot\|Y'\|} - 1\right) \\ &\quad\cdot(\|X'-Y'\|^2 - (\|X'\|-\|Y'\|)^2) \\ &\le \|X-Y\|^2 \qquad (28) \end{aligned}$$

where (27) follows because

$$(\alpha(X) - \alpha(Y))^2 \le (x_k - y_k)^2 + (\|X'\| - \|Y'\|)^2.$$

To see this, note that $\alpha(Y)$ depends only on y_k, and not any of $y_1, \cdots, y_{k-1}$. Hence, for the computation of $\alpha(Y)^2$, no generality is lost by assuming $(y_1, \cdots, y_{k-1})$ is a constant times $(x_1, \cdots, x_{k-1})$. This implies $X_L = Y_L$ and

$$\|X' - Y'\|^2 = (\|X'\| - \|Y'\|)^2.$$

Thus

$$\begin{aligned}
&(\alpha(X) - \alpha(Y))^2\\
&= [(\|(X_L)'\| - \|X_L - X\|) - (\|(X_L)'\| - \|X_L - Y\|)]^2\\
&= (\|X_L - Y\| - \|X_L - X\|)^2\\
&\le \|X - Y\|^2 \qquad (29)\\
&= (x_k - y_k)^2 + \|X' - Y'\|^2\\
&= (x_k - y_k)^2 + (\|X'\| - \|Y'\|)^2
\end{aligned}$$

where (29) follows by the triangle inequality. Also, (28) follows since

$$\begin{aligned}
\alpha(X) &= \|(X_L)'\| - \|X_L - X\|\\
&\le \|(X_L)'\| - \|(X_L)' - X'\| = \|X'\|
\end{aligned}$$

(and similarly $\alpha(Y) \le \|Y'\|$) and

$$\|X' - Y'\|^2 - (\|X'\| - \|Y'\|)^2 \ge 0$$

by the Cauchy–Schwarz inequality.

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